Investment under Uncertainty: The Role of Inventory Dynamics

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Abstract

Inventories of finished goods is very common under market uncertainty. We build a continuous model to study how the inventory will impact firm value and investment decisions. Our model shows that the value of a company following the optimal inventory policy can be significantly higher than the traditional non-inventory company, particularly if the inventory-holding cost is not large. This premium becomes small as holding cost is increased, and large when demand is volatile, and when price elasticity is large. We also show that the optimal investment size can be significantly larger than the traditional no-inventory firm, particularly when the inventory-holding cost is low, demand volatility is high, and price elasticity is low. This paper develops a simulation algorithm to solve iterative optimization problem in a path-dependent economy.

Key Words: inventory dynamics, real options, corporate investment

1. Introduction

A large number of papers have used the contingent-claim model of a firm for valuation and investment/financing decisions under uncertainty. In these papers, uncertainty is introduced by means of a stochastic underlying variable such as revenue or output price or demand strength, which follows an exogenously specified random process. The firm valuation and the investment/capital structure decisions are based on this uncertain state variable.

These papers all assume that the firm sells its output when produced and there is no possibility of maintaining any inventory; thus, they ignore any effect of inventory management on firm performance or value. Examples include Huberts et al. (2015), Jou and Lee (2008), Mauer and Ott (2000), Miao (2005), and many others. However, available empirical evidence indicates that inventory management does in fact have a significant impact on the firm's performance, e.g., Basu and Wang (2011), Elsayed (2015), Koumanakos (2008), Kroes and Manikas (2018), and Ndubuisi et al. (2020). In this paper, therefore, we address the question: how does (optimal) inventory management impact the value of the firm?

There have been a handful of papers that include inventory, but these papers focus on raw-material inventory or work-in-process inventory and not on output inventory, which is the focus of our paper. For instance, Bianco and Gamba (2019) input inventory use as an operational hedge for risk management, to mitigate the price risk of input materials. Cortazar and Schwartz (1993) examine the valuation of a company with a two-step manufacturing process with work-in-process inventory.

We examine a firm that has the ability to maintain inventory of the output, and compare it to the traditional model's no-inventory firm. We focus on how much this ability to maintain inventory is worth, and how this premium in value (relative to the no-inventory firm) is affected by various economic parameters. We consider both cases, i.e., when the firm chooses an inventory policy so as to maximize current profit or firm value (since both seem to be used in practice).

The main results are as follows. First, if inventory holding cost is small, then the firm value with inventory can be substantially larger than in the traditional (no-inventory) models. Second, the behavior of firm value with respect to investment size is quite different for firm with inventory and without inventory; hence the conclusions of the traditional literature regarding investment size might have to be modified for the realistic situation of a firm with inventory Third, if an inventory-carrying firm's inventory policy is based on maximizing profits rather than value, it might end up with a firm value that is below the benchmark (no-inventory) firm, particularly if the demand elasticity is large and demand level is small; thus, maximizing firm profits might end up causing destruction of firm value.

The rest of the paper is organized as follows. Section 2 describes and derives the model with and without the ability to maintain inventory. Section 3 presents the results of the model. Section 4 summarizes and concludes.

2. The Model

A firm has a production facility whose size is given by the amount of capital K. This facility allows it to produce Q units of the output per unit time, where Q is given by $Q = K^{\delta}$; the exponent δ can be viewed as the returns-to-scale of the technology used. The output is sold in the product market, at a price given by:

$$p = \theta - \gamma q, \tag{1}$$

where q is the amount sold, θ is the random/stochastic strength of demand (or demand shock), and γ is the sensitivity of the price to the amount sold (or the elasticity of demand). This price process is commonly used in the literature to represent the output's demand curve (Aguerrevere, 2003, Dangl, 1999, Huberts et al., 2015). The strength of demand θ introduces uncertainty in the model, and is assumed to follow the lognormal process:

$$d\theta = \mu\theta dt + \sigma\theta dZ, \tag{2}$$

where μ and σ are the trend and volatility of the demand process and Z is a standard Wiener Process.

We assume that the plant always operates at full capacity, that is, the production rate is always Q, as in Bar-Ilan and Strange (1999), Dangl (1999), etc. This is partly because of analytical tractability; if the firm could vary the output rate, it would make the analysis much more complicated. However, this is also a common modeling assumption, because it is a reasonable description of many real-world process industries such as paper, chemicals, etc. (Lederer and Mehta, 2005). Moreover, it is consistent with the "price postponement with clearance" argument of Van Mieghem and Dada (1999). Finally, in many industries the firms make the production plans before the actual realization of market demand (Della Seta et al., 2012), and many firms find it difficult to produce below full capacity because of commitments to suppliers and because of fixed costs associated with flexibility (Goyal and Netessine, 2007).

Finally, the cost of investing in the production plant has both a constant component and a component that is increasing linearly in the investment size (amount of capital, K). Let this investment cost be given by $I(Q) = m_0 + m_1K = m_0 + m_1Q^{1/\delta}$.

Since we are studying the effect of maintaining output inventory, we study both cases – the firm with and without inventory. We call the latter the "inflexible" or "benchmark" firm as it is this type of firm that has been examined in the literature so far. Also, a firm with the ability to maintain inventory might have one of two objectives when making its inventory decisions; its objective could be to maximize either the firm's profit for the period or the value of the firm. A quick look at the Production

Economics literature indicates that in many (if nor most) cases the objective is specified as profit-maximization; on the other hand, in the Finance literature the objective is value-maximization. We therefore examine both cases for the inventory-holding firm.

For high price-sensitivity γ, the profit-maximizing firm might sell a smaller amount to keep price high (so as to maximize current profit); however, this will increase future inventory level and drive up inventory costs and thereby reduce firm value. Since it acts myopically in ignoring future costs when making decisions, we call the profit-maximizing firm a "myopic" firm. The value-maximizing firm, on the other hand, behaves strategically in considering the overall effect of its decisions on the value of the firm (taking into account *all* costs), hence we call it a "strategic" firm. Thus, we analyze three different firms – (i) benchmark firm (no inventory), (ii) myopic firm (with inventory), and (iii) strategic firm (with inventory).

2.1. Benchmark Firm Valuation

The benchmark firm produces and sells at a rate of Q units per unit time. Then, its profit flow is given by:

$$\pi(\theta) = pq - cq = (\theta - \gamma q)q - cq = Q\theta - Q(\gamma Q + c) \tag{3}$$

Then, the value of the plant is given by:

$$V(\theta) = \mathbb{E}_0^{\mathbb{Q}} \int_0^T e^{-r\tau} \pi_\tau d\tau = \frac{1-e^{-(r-\mu)T}}{r-\mu} \theta Q - \frac{1-e^{-rT}}{r} Q(\gamma Q + c), \tag{4}$$

where T is the remaining life of the project.

This is the standard approach in the literature, and the above valuation is consistent with the existing models, e.g., Bar-Ilan and Strange (1999). If the firm also chooses the size of the investment optimally, then it will maximize the above firm value at the time of investment less the investment cost, i.e., $Q^* = \max_{Q}\{V(\theta_0, Q) - (m_0 + m_1 Q^{1/\delta})\}$, where θ_0 is the demand strength at the time of investment, and m_0 is the fixed investment cost and m_1 is the variable cost of investment per unit of capital (recall that capital is $K = Q^{1/\delta}$).

2.2. Myopic Firm Valuation

Suppose the existing inventory level is N and the inventory holding cost is \$k per unit of product per unit time. Then the profit stream is given by:

$$\pi(\theta) \; = \; pq - cQ - kN \; = \; (\theta - \gamma q)q - cQ - kN \tag{5}$$

To maximize the instantaneous profit, it will set $d\pi/dq = 0$, which gives:

$$\theta - 2\gamma q^* - k\frac{dN}{dq} = 0, \text{ or } q^* = \frac{\theta - k\frac{dN}{dq}}{2\gamma} \tag{6}$$

We know that when one more unit is sold (i.e., q is up by 1), the inventory balance will decline by 1 (i.e., N will fall by 1), hence $\frac{dN}{dq} = -1$. This gives us the optimal amount the myopic firm will sell at any instant:

$$q^* = \frac{\theta + k}{2\gamma} \tag{7}$$

However, there is an upper limit on how many units the firm can sell at any point in time; suppose it can draw down inventory at a rate of $\dot{N}$ units per unit time. Then the sales amount is limited by $q \leq (Q + \dot{N})$. The demand threshold where the sales amount reaches the upper limit is given by $\overline{\theta} = 2\gamma(Q + \dot{N}) - k$.

The valuation of the myopic firm cannot be expressed analytically, but the general approach is as follows: firm value $V(t,\theta)$ is given by:

$$V(t,\theta) = \mathbb{E}_t^{\mathbb{Q}} \int_t^{\infty} e^{-r\tau} \pi_\tau(\tau, \theta | q_\tau^*) d\tau$$

Under *Feynman–Kac*, a PDE can be written as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\theta^2 \frac{\partial^2 V}{\partial \theta^2} + \mu\theta \frac{\partial V}{\partial \theta} + \pi(q_t^*) = rV \tag{8}$$

Which subjects to boundary conditions

$$V(\theta \downarrow 0, t) = 0 \tag{9}$$

When θ is very large, the firm will always sale at full and no inventory

$$V(\theta \uparrow \infty, t) = Q\left(\frac{\theta}{r-\mu} - \frac{\gamma Q + c}{r}\right) \tag{10}$$

Suppose there is a life limit t = T such that firm will be liquidated with only inventory left with discounted price, same as storage cost

$$V(\theta, T) = 0 \tag{11}$$

Unfortunately, the PDE still couldn't be solved since it is path-dependent, that is, current profit depends on previous inventory history, we will turn to Monte Carlo simulation to solve this, and the numerical details are at Appendix A.

2.3. Strategic Firm Valuation

In this case the key assumption is that: firms have zero knowledge about instantaneous optimal expected sales and always produce at full capacity. We admit it is a strongest assumption and firms may or may not sell all of the output, and the output price depends on the real sales. Future research is needed when firms know some information about optimal sales and don't need to produce at full capacity all the time. For example, a Bayesian's learning experience can be incorporated.

In this case the instant sales q^* could be larger than capacity Q due to the inventory

the profit function will be

$$\pi_t = (\theta - \gamma q_t)q_t - cQ - kN_t \tag{12}$$

here k is the storage cost of inventories. Note that the optimal q is a function of θ and N, e.g. $q^* \equiv q^*(\theta, N)$. The optimal sales q^* is hard to express explicitly since it is a function of stochastic variable inventory, which in turn varies with q^*. We call it strategic firm since the instant q^* for all t (0,1, …, T) will be optimized to max firm value instead of instant profit

The inventory dynamics are as follows

$$\frac{dN}{dt} = Q - q_t \tag{13}$$

The value of firm after investment, $V(t, \theta, N)$ is governed by following

$$V(t, \theta) = \max_{q^*}\left[E_t^Q \int_t^\infty e^{-r\tau}\pi_\tau(\tau, \theta, N)d\tau\right]$$

The above can be rewritten as following PDE under *Feynman–Kac*.

$$\max_{q^*}\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\theta^2\frac{\partial^2 V}{\partial \theta^2} + \mu\theta\frac{\partial V}{\partial \theta} + (Q - q)\frac{\partial V}{\partial N} + \pi(\theta, N|q)\right] = rV \tag{14}$$

This a 3-D PDE and let's reduce to 2-D by replacing inventory N. Notice that from the inventory dynamic we can solve its value up to time t

$$N_t = tQ - \int_0^t q_\tau d\tau$$

And instant profit becomes

$$\pi_t = (\theta - \gamma q_t)q_t - cQ - ktQ - k\int_0^t q_\tau d\tau \tag{15}$$

The valuation can be rewritten as

$$V(t,\theta) = \max_{q^*}\left[E_t^Q \int_t^\infty e^{-r\tau}\pi_\tau(\tau,\theta)d\tau\right]$$

A 2-D PDE can be written as

$$\max_{q^*}\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\theta^2\frac{\partial^2 V}{\partial\theta^2} + \mu\theta\frac{\partial V}{\partial\theta} + \pi(\theta, t|q)\right] = rV \tag{16}$$

In a finite difference method (FDM), you need to backward from t = T to 0, however, the profit flows depends on previous optimal q^* (to calculate inventory), that also has to be solved in all PDE domain. However, such difficulty will be largely mitigated in myopic firm since we can solve profit for all domain before implement FDM. It has similar boundary conditions as for myopic firms:

When demand approaches zero, the firm value becomes zero

$$V(\theta \downarrow 0, t) = 0 \tag{17}$$

When θ is very large, the firm will always sale at full and no inventory is generated

$$V(\theta \uparrow \infty, t) = Q\left(\frac{\theta}{r-\mu} - \frac{\gamma Q + c}{r}\right) \tag{18}$$

Suppose there is a life limit t = T such that firm will be liquidated with zero

$$V(\theta, T) = 0 \tag{19}$$

Before investment, e.g., at $t = 0$, there is no inventories, and the investment option with infinite maturity can be valued under classic

$$F(\theta) = A\theta^a \tag{20}$$

Our aim is to solve optimal investment timing and capacity to maximize A

$$A\theta^{a} = V(0, \theta) - I(K) \tag{21}$$

***A brief discussion on the numeric methods*:**

The model can't be solved in a classic numerical method such as finite difference method due to the path-dependent nature of inventory dynamics. The reason is that our path-dependence is far more complex than financial options such as Asian and Lookback and it has to satisfy a loop on time series: (1) the firm's current (optimal) inventory is an outcome of past production and optimal sales (2) the past optimal sales link to the current and future (optimal) inventories. Therefore we have to turn to Monte Carlo simulation to solve it. However simulation is costly on computation time, particularly for the strategic-type firm.

In particular, a longer time series (e.g. a longer firm life) will exponentially increase CPU time. For example, if we simply search for optimal sales in each time spot to maximize firm value at entry, it may cost several minutes to half of an hour to find the solution for a short firm life and for a single path! Such method can't solve the problem for a large batch of simulation paths (say, 10,000 paths). Under our novel searching algorithm (see Appendix B), the CPU time to calculate a path can be decreased to 0.01 second, which is still very time-consuming, particularly for comparative static analysis or searching for optimal investment decisions, thus we have to limit to max firm life of 10 years with 1 year time step. To enhance the accuracy, we then simulation 10 batches of simulation to get the standard error at 1% level.

2.4. Analysis

Before presenting the numerical results, we discuss briefly what we can expect from the computations. As discussed above, all firms will produce at the rate of Q, but will sell different amounts. The inflexible (benchmark) firm does not have the ability to maintain inventory, hence it will sell all that is produced (Q). Both the myopic and the strategic firm can sell at a different rate, since they have the ability to maintain inventory; therefore, they will sell more than or less than the amount they produce (Q).[1] However, their objective will be different – the myopic firm wants to maximize profit every period, while the strategic firm wants to maximize firm value. Since its decisions are designed to maximize firm value, it is clear that the strategic firm's value will be the highest of the three types; the next should be the myopic firm, since it also has flexibility regarding sales but uses its flexibility to myopically maximize instantaneous profits. The inflexible (benchmark) firm should have the lowest value, since it is not able to maintain inventory and hence has no flexibility regarding sales. Thus, the usual ordering of firm value is strategic, myopic and benchmark, in order of decreasing value. There is, however, one exception, discussed below.

If the myopic firm sells a smaller quantity (i.e., q^* is smaller), then it will build up more inventory, which might destroy firm value because of inventory costs (recall that the myopic firm is maximizing profits, not firm value). Therefore, because of the excess inventory carrying cost, it is possible (although perhaps counter-intuitive) that, when q^* is for small enough, the value of the myopic firm falls below the value of the benchmark firm. Recall that the myopic sells the output at the rate of $q^* = \frac{\theta+k}{2\gamma}$; hence q^* is increasing in θ and k, while it is decreasing in the demand elasticity γ. Therefore, the myopic firm value is more likely to be below the benchmark value when θ and k are small and when γ is large; alternatively, the myopic premium is less likely to be negative when demand level (θ) is large and demand elasticity (γ) is small. Both of these are confirmed by our numerical results below.

Thus, the premium for the strategic firm value over benchmark firm will always be positive. However, the value of the ability to keep inventory will decline as the inventory holding cost rises, thus

[1] Note that it can sell more than it produces only if there is inventory on hand.

the premium should be decreasing in k (and approaching zero for large enough k). On the other hand, the premium for the myopic firm over the benchmark firm can be positive or negative. Moreover, for all cases (positive or negative), this premium should approach zero for large enough k, since the ability to maintain inventory will make no difference if inventory holding costs are very high.

3. Numerical Results from the Model

3.1. Base-case parameter values

In this section, we present numerical results from the simulation. We start with a comparison of firm value as a function of inventory holding cost k, for the myopic firm and the strategic firm as well as the benchmark firm (as a benchmark, since it will obviously be independent of k). For numerical results, we need to specify the input variables. We use a set of "base case" values for the input variables, as follows: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0.01$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0$, $m_1 = 5$, $K = 8$, $T = 10$, $k = 0.2$, and $\theta_0 = 6$. We choose them based on following economic literature:

For the interest rate, we adopted r = 4% as in Jeanneret (2017) and Aretz and Pope (2018), in line with the average 10-year U.S. treasury rate. For the expected drift of output price, we use $\mu = 1\%$, conforming to Tserlukevich (2008), Lyandres and Zhdanov (2013) and Arnold (2014). The cash flow volatility σ follows Lyandres and Zhdanov (2013) and Arnold (2014), which stands at 0.2. The return-to-scale parameter δ is set to 0.5, the approximate average of past estimates. [2] All others are structural parameters and can be studied in a comparative statics.

3.2. Valuation and Premium over Benchmark Firm

3.2.1. Base case results

[2] The return to scale varies in a large range in the past papers to entertain corresponding calibration performance. However, varying its value does not alter our main conclusion, so we take an approximately average of past values, to name a few, they are Zhang (2005, $\gamma = 0.3$), Miao (2005, $\gamma = 0.4$), Danis and Gamba (2018, $\gamma = 0.475$), Nikolov *et al*. (2018, $\gamma = 0.66$), Riddick and Whited (2009, $\gamma = 0.75$)

Figure 1(a) shows the firm value as a function of inventory holding cost k for the three firms: benchmark or inflexible firm (black line), myopic firm (blue line) and strategic firm (broken red line), and Figure 1(b) shows the same results in terms of premium over benchmark firm value. As expected from the above discussion, the benchmark firm value is independent of k and the strategic firm value is the highest over the entire range. Also, the value of the strategic firm is a decreasing function of k; for small k, the difference in firm value is substantial (about 11% above benchmark firm value for $k = 0$); but the difference falls rapidly with k, and it becomes negligible levels for k exceeding 1.5. This is not surprising since inventory becomes more expensive to support as k is increased, hence the ability to maintain inventory becomes less valuable; thus, the strategic firm value (and premium) is decreasing in k.

The behaviour of the myopic firm value is a little different. Firstly, it is a U-shaped function of k, falling from 6.75% for $k = 0$ to –12.1% for $k = 2$, and then rising slightly as k is increased further. Secondly, the myopic firm value falls below the benchmark firm value when k is large enough (for k exceeding 0.4).

Thus, for small inventory holding cost, the myopic firm value can also be significantly higher than the benchmark firm value. However, for larger inventory holding cost, the ability to maintain inventory can destroy value if it is not utilized optimally, e.g., if maximizing short-term profit instead of value as in the myopic firm; in the base case, up to 12% of firm value can be destroyed this way. The U-shaped relationship can be explained as follows. As k is increased from 0, there are two opposing effects on firm value: (i) direct effect: a higher k means higher inventory holding cost, which lowers the firm value and thus results in a downward-sloping curve; and (ii) indirect effect through q^*: as discussed above, a higher k results in higher q^*, that is, the firm starts reducing inventory; the resulting lower inventory holding cost will increase firm value and this will result in an upward-sloping curve. The latter effect dominates for larger k, giving rise to the overall U-shaped relationship between inventory holding cost and myopic firm value observed in Figure 1.

Figure 1 about here

There are two points worth noting in these results. First, the benefits of maintaining inventory diminish rapidly as the inventory holding cost rises. Therefore, in industries where the inventory holding costs are high, it would make sense to use more inventory-minimizing techniques such as JIT (just in time). Second, an inventory management policy of choosing inventory levels based on maximizing profits (as in the myopic firm) could, in fact, end up reducing firm value relative to an inflexible (a no-inventory) firm. This is an important point because, in practice, inventory policy is often implemented out by operating managers whose objective is to maximize annual profits (Bassamboo et al., 2020, Canyakmaz et al., 2021, Li et al., 2021, Ma et al., 2021, Transchel et al., 2021, Zhao, 2008). Our result indicates that profit-maximizing might not, in fact, be optimal because it could result in reduced firm value (particularly for large inventory holding cost). This suggests that it is better for inventory policy to be set on the basis of firm value rather than profits.

Figure 2 about here

Next, recall from our discussion above that the myopic firm's premium is less likely to be negative when demand level (θ) is large and demand elasticity (γ) is small, since q^* will be larger and the firm will be less likely to accumulate large quantities of inventory, and with lower inventory holding costs there will be less value destruction. Thus, for high θ and/or low γ, we would expect the negative premium in myopic firm value to be smaller than in the base case. Repeating the numerical computations with these two scenarios ($\theta = 12$ and $\gamma = 0.5$), we find that this is indeed the case. As Figure 2 shows, in both cases, the negative premium that we noted in Figure 1(b) becomes much smaller;

in fact, the premium is very close to zero. This is because, with larger q^*, the myopic firm will be selling more and leaving less in inventory, making it resemble more and more the benchmark firm; thus, with high θ and low γ, the difference between myopic and benchmark firms will shrink significantly and the premium will be closer to zero, consistent with Figure 2.

3.2.2. Comparative Statics

Since there is no inventory with the benchmark firm, it has to sell everything it produces, even it means selling at low prices during low-demand periods. On the other hand, both the myopic and the strategic firm can maintain inventory rather than selling the output at low prices, the former doing it in such a way as to maximize (short-term) profit while the latter maximizes firm value. Thus, as explained in Section ??, both firms will generally be valued at a premium over the benchmark firm (although there might be exceptions for the myopic firm, as explained above). In this section, we find that this is indeed the case; the numerical results show that, for reasonable parameter values, the premium over the no-inventory firm is positive for both firms (Figure 3(a) to (g)).

We now look at how the various input parameters affect the premium for the myopic and strategic firms over the value of the benchmark no-inventory firm. Anything that reduces the need to maintain inventory (thereby reducing the value of the option to maintain inventory), e.g., greater demand level, will move the premium closer to zero. Conversely, anything that makes the option more important (e.g., higher demand volatility) will move the premium away from zero.[3] This is indeed what our numerical results (below) confirm.

<u>Figure 3 about here</u>

[3] Recall that in the base case (Figure 1), the premium for the myopic firm could be positive or negative depending on the inventory holding cost k. For positive (negative) premium, moving closer to zero implies that the premium will be decreasing (increasing); moving away from zero will imply just the opposite. Therefore, for the myopic firm, whether the premium is increasing or decreasing will depend on whether it is positive or negative. This is illustrated in Figure 4.

First, we look at the effect of demand volatility (σ). For both strategic and myopic firms, the premium is an increasing function of σ. As we know from standard option theory, greater volatility increases option values. Since both firms have the option to maintain inventory, it is not surprising that the firm value (and premium over benchmark firm) is increasing in volatility in both cases, as shown in Figure 3(a).

Next, a higher demand growth rate (μ) has a negative effect on both strategic and myopic firm value, as shown in Figure 3(b), and both premiums approach zero when the growth rate is very large. This is also as expected, since a higher growth rate will cause both firms to sell more, hence inventory will play a diminished role and the premium will trend towards zero as a result (and they are decreasing functions of μ since both premiums are positive).

A larger capacity (K) means the firm is producing more, since it always produces at full capacity; when it is producing more, there will be more inventory, hence inventory will play a larger role. Thus, the magnitude of the premium (resulting from the ability to maintain inventory) will rise. As a result, premiums will move away from zero as K is increased, i.e., since premium is positive it will increase with K in both cases. As we see in Figure 3(c), this is indeed the case.

Next, greater demand price sensitivity (γ) will result in a lower current output price (all else remaining unchanged), hence both flexible companies will sell less and therefore maintain higher inventory levels. This means that a higher γ will cause the inventory effect to be larger; hence the magnitude of the premium will be larger (i.e., it will move away from zero) as γ is increased. That is, the premium, if positive (negative), will be increasing (decreasing) in price sensitivity. As we see in Figure 3(d), this is indeed the case, with the premium for both firms being positive and increasing in γ.

For the operating cost (c), note that a higher c means the margin is lower, which has the same effect as a lower price or a higher γ; thus, as in the case of γ, a higher c will result in inventory becoming more important, hence the magnitude of the premium will increase with c. That is, the premium will move away from zero as c is increased, or the premium, if positive (negative), will be increasing (decreasing) in c. As shown in Figure 3(e), this is exactly what we find, with the premium in both cases being positive and increasing in c.

A larger returns-to-scale parameter (δ) implies that a greater quantity will be produced with the same amount of capital, therefore the ability to maintain inventory will be more valuable. Thus, the premium should be moving away from zero (if positive, an increasing function of δ), which is consistent with our numerical results shown in Figure 3(f).

When the demand level (θ_0) is higher, both firms will sell more and thus have less in inventory; therefore, inventory will play a reduced role and the magnitude of the premium will fall, i.e., the premium will move towards zero. As a result, a positive (negative) premium will be a decreasing (decreasing) function of θ_0. This is consistent with our numerical results, as shown in Figure 3(g). Finally, the two parameters interest rate (r) and investment cost (m_1) do not have a noticeable effect on the two premiums, because the myopic/strategic firm is impacted the same way as the benchmark firm.

<u>Figure 4 about here</u>

In the above comparative static results, the myopic firm premium was positive in all cases; with negative premium, the relationship will appear somewhat difference. To illustrate, we show in Figure 4 the effect of μ and γ when the myopic premium is negative (by setting inventory-holding cost $k = 0.5$ instead of 0.2). We note that (for the myopic firm premium) the effect is now different from the previous case (Figure 3), i.e., increasing in μ and decreasing in γ; this is because the myopic premium is negative, as discussed above.

To summarize, the above comparative static results show that the premium over the benchmark firm value can vary a lot when the input parameters are varied. Moreover, we also find (not shown) that the response of firm value to certain parameter values (e.g., investment size) can be quite different for firms with inventory and without inventory; the implication is that investment size and its sensitivity to various parameters can be quite different from the traditional models if we include the ability to maintain inventory.

3.3. Optimal Investment Size

An important component of the corporate investment decision is the investment size. The ability to maintain inventory should have an impact on the investment size (or production capacity) decision. The benchmark firm must sell everything it produces right away, hence it will be more hesitant to invest in large capacity relative to the myopic firm or the strategic firm. Thus, the optimal investment size should be higher when the firm has the inventory option.

Figure 5 shows the firm value for all three cases as a function of investment size K. We note that the optimal K for the myopic and strategic firm is significantly larger than that of the myopic firm (9.4 versus 7.6 units of capital), as expected from the above discussion; note that there is no difference in optimal size between the myopic and strategic firm. Clearly, the investment size can be significantly impacted by the ability to maintain output inventory.

<u>Figures 5 and 6 about here</u>

We also take a look at how the optimal investment size is affected by certain parameter values, and whether the inventory option affects this relationship. In particular we focus on the following parameters: inventory holding cost k, demand volatility σ and demand elasticity γ. These results are shown in Figure 6. Not surprisingly, the benchmark firm's K^* is unaffected by k, while K^* for both myopic and strategic firm are decreasing in k. The strategic firm's K^* converges to that of the benchmark firm as k is increases sufficiently; however, the myopic firm's K^* can be smaller than that of the benchmark firm for high enough k. As discussed in Section 3.2, the myopic firm value can be below the benchmark value if k is high enough because it maximizes profit instead of value. For the same reason, the myopic firm will choose a smaller size than benchmark firm for large enough k, as shown in Figure 6(a).

Next, as Figure 6(b) shows, the benchmark K^* is independent of σ; this is because it has no embedded options, hence the decision is unaffected by volatility. On the other hand, both myopic and strategic firm have the inventory option, hence the value increases with volatility, thus K^* in both cases is increasing in σ; we also note in Figure 6(b) that the optimal size is virtually identical for myopic and strategic firm for all σ. Finally, Figure 6(c) shows the effect of price elasticity γ. A larger γ means greater output will be relative less attractive because the price decline will be larger. It is then not surprising that K^* is decreasing in γ for all three firms. Once again, there is no difference in K^* for the myopic and strategic firms. Overall, it seems that there is no difference between optimal investment size for myopic firm and strategic firm, although there might be substantial differences in the valuation of the two.

4. Conclusion

This paper studies the contingent-claim valuation of a company that can maintain an inventory of its output. Existing models ignore the possibility inventory. We show that the value of a company following the optimal inventory policy can be significantly higher than the traditional non-inventory company, particularly if the inventory-holding cost is not large. This premium shrinks as holding cost is increased, and disappears for large enough holding cost, and is particularly large when demand is volatile, when demand level is low, and when price elasticity is large. Thus, the ability to maintain inventory could potentially have a significant impact on the company's investment and financing decisions.

It is also shown that, when the company's inventory policy is set myopically so as to maximize the current profit rather than long-term value (as is often the case in practice), it is possible that firm value actually falls below the traditional no-inventory firm, i.e., the premium turns negative. We also show that the optimal investment size can be significantly larger than the traditional no-inventory firm, particularly when the inventory-holding cost is low, demand volatility is high, and price elasticity is low.

We are currently working on extending our results by examining the average inventory levels and how they are impacted by various economic parameters.

References

Aretz, K and P. Peter (2018) Real Options Models of the Firm, Capacity Overhang, and the Cross Section of Stock Returns. *Journal of Finance*, 73(3), 1363-1415

Bassamboo, A., A. Moreno and I. Stamatopoulos (2020), Inventory Auditing and Replenishment Using Point-of-Sales Data, *Production and Operations Management* 29, 1219-1231.

Basu, N. and X. Wang (2011) Evidence on the Relation between Inventory Changes, Earnings and Firm Value, *International Journal of Business and Finance Research* 5, 1-14.

Bianco, M. and A. Gamba (2019) Inventory and Corporate Risk Management, *Review of Corporate Finance Studies* 8, 97-145.

Canyakmaz, C., S. Ozekici and F. Karaesmen (2021) A Newsvendor Problem with Markup Pricing in the Presence of Within-period Price Fluctuations, *European Journal of Operational Research*, doi: https://doi.org/10.1016/j.ejor.2021.09.042.

Cortazar, G. and E.S. Schwartz (1993) A Compound Option Model of Production and Intermediate Inventories, *Journal of Business* 66, 517-540.

Elsayed, K. (2015) Exploring the Relation between Efficiency of Inventory Management and Firm Performance: An Empirical Research, *International Journal of Services and Operations Management* 21, 73-86.

Goyal, M. and S. Netessine (2007) Strategic Technology Choice and Capacity Investment under Demand Uncertainty, *Management Science* 53, 192-207.

Hubert, N.F.D., K.J.M. Huisman, P.M. Kort and M.N. Lavrutich (2015) Capacity Choice in (Strategic) Real Options Models: A Survey, *Dynamic Games and Applications* 5, 424-439.

Jou, J-B., and T. Lee (2008) Irreversible investment, financing, and bankruptcy decisions in an oligopoly, *Journal of Financial and Quantitative Analysis* 43, 769–786.

Koumanakos, D.P. (2008) The Effect of Inventory Management on Firm Performance, *International Journal of Productivity and Performance Management* 57, 355-369.

Kroes, J.R. and A.S. Manikas (2018) An Exploration of ‘Sticky’ Inventory Management in the Manufacturing Industry, *Production and Planning Control* 29, 131-142.

Li, D., J. Chen and Y. Liao (2021) Optimal Decisions on Prices, Order Quantities, and Returns Policies in a Supply Chain with Two-period Selling, *European Journal of Operational Research* 290, 1063-82.

Lyandres, E., and Zhdanov, A. (2013). Investment opportunities and bankruptcy prediction. Journal of Financial Markets, 16(3), 439-476.

Ma, S., Z. Jemaiand Q. Bai (2021) Optimal Pricing and Ordering Decisions for a Retailer Using Multiple Discounts, *European Journal of Operational Research*, doi: https://doi.org/10.1016/j.ejor.2021.10.004.

Mauer, D.C. and S. Ott (2000) Agency Costs, Under-investment, and Optimal Capital Structure: The Effect of Growth Options to Expand, in: Brennan, M. and Trigeorgis, L. (Eds.), *Project Flexibility, Agency, and Competition*, Oxford University Press, New York, 151-180.

Miao, J. (2005) Optimal Capital Structure and Industry Dynamics, *Journal of Finance* 60, 2621–2659.

Ndubuisi, A.N., O.J. Chinyere, E. Beatrice O. and E.P. Uche (2020) Inventory Management and Firm Performance: Evidence from Brewery Firms Listed on Nigeria Stock Exchange, *International Journal of Research in Business, Economics and Management* 2, 72-93.

Transchel, S., M.E. Buisman and R. Haijema (2021) Joint Assortment and Inventory Optimization for Vertically Differentiated Products under Consumer-driven Substitution, *European Journal of Operational Research*, doi: https://doi.org/10.1016/j.ejor.2021.09.041.

Tserlukevich, S. (2008) Can real options explain financing behavior? Journal Financial Economics, 89, 232-252

Van Mieghem, J.A. and M. Dada (1999) Price versus Production Postponement: Capacity and Competition, *Management Science* 45, 1631-1649.

Zhao, X. (2008) Coordinating a Supply Chain System with Retailers Under Both Price and Inventory Competition, *Production and Operations Management* 17, 532-542.

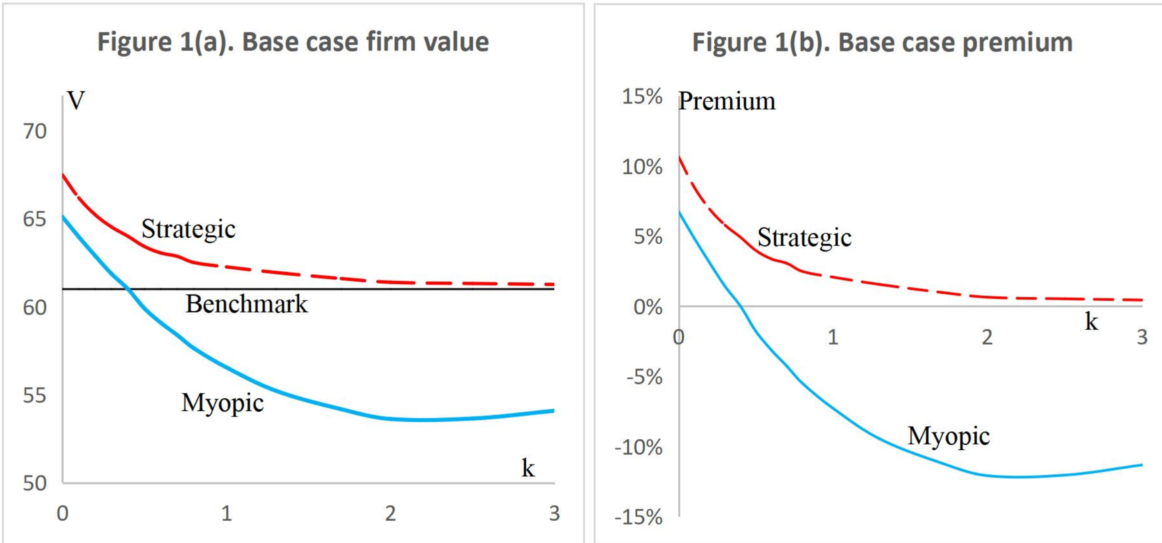


Figure 1. Shows the firm value for the three firm types (benchmark, myopic and strategic) and the premium over benchmark for the myopic and strategic firm. The base-case parameter values are used: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0$, $m_1 = 5$, $K = 8$, $T = 10$, $\theta_0 = 2$, and $\theta_0 = 6$.

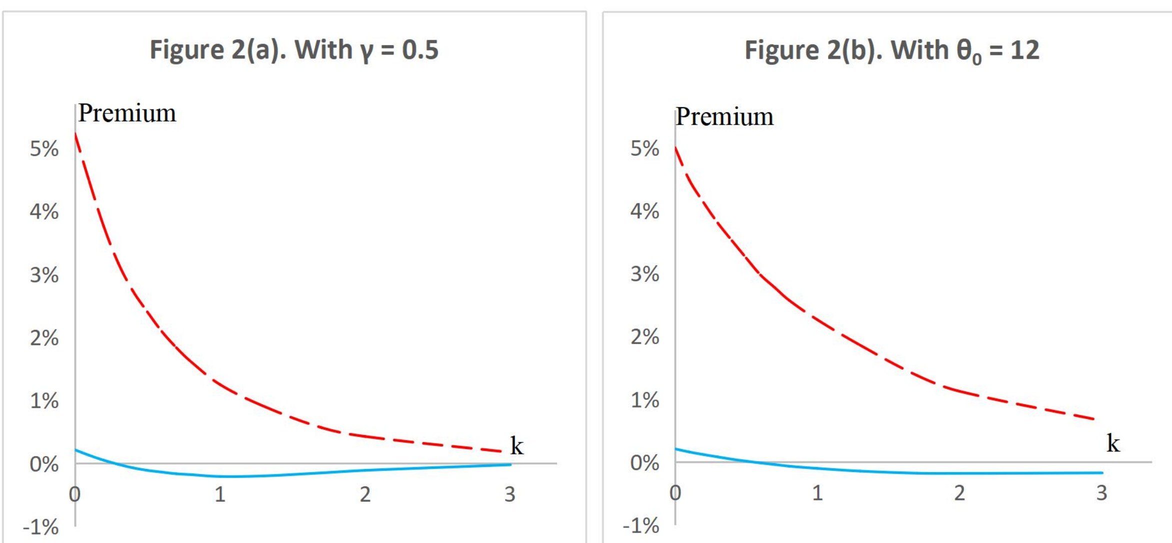


Figure 2. Shows the premium over the benchmark firm for two special cases, $\gamma = 0.5$ and $\theta_0 = 12$ (base-case value for all other parameters). The broken red line shows the strategic firm, and the solid blue line shows the myopic firm. Apart from the above parameters, the base-case values are used: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0$, $m_1 = 5$, $K = 8$, $T = 10$, $\theta_0 = 2$, $k = 0.2$, and $\theta_0 = 6$.

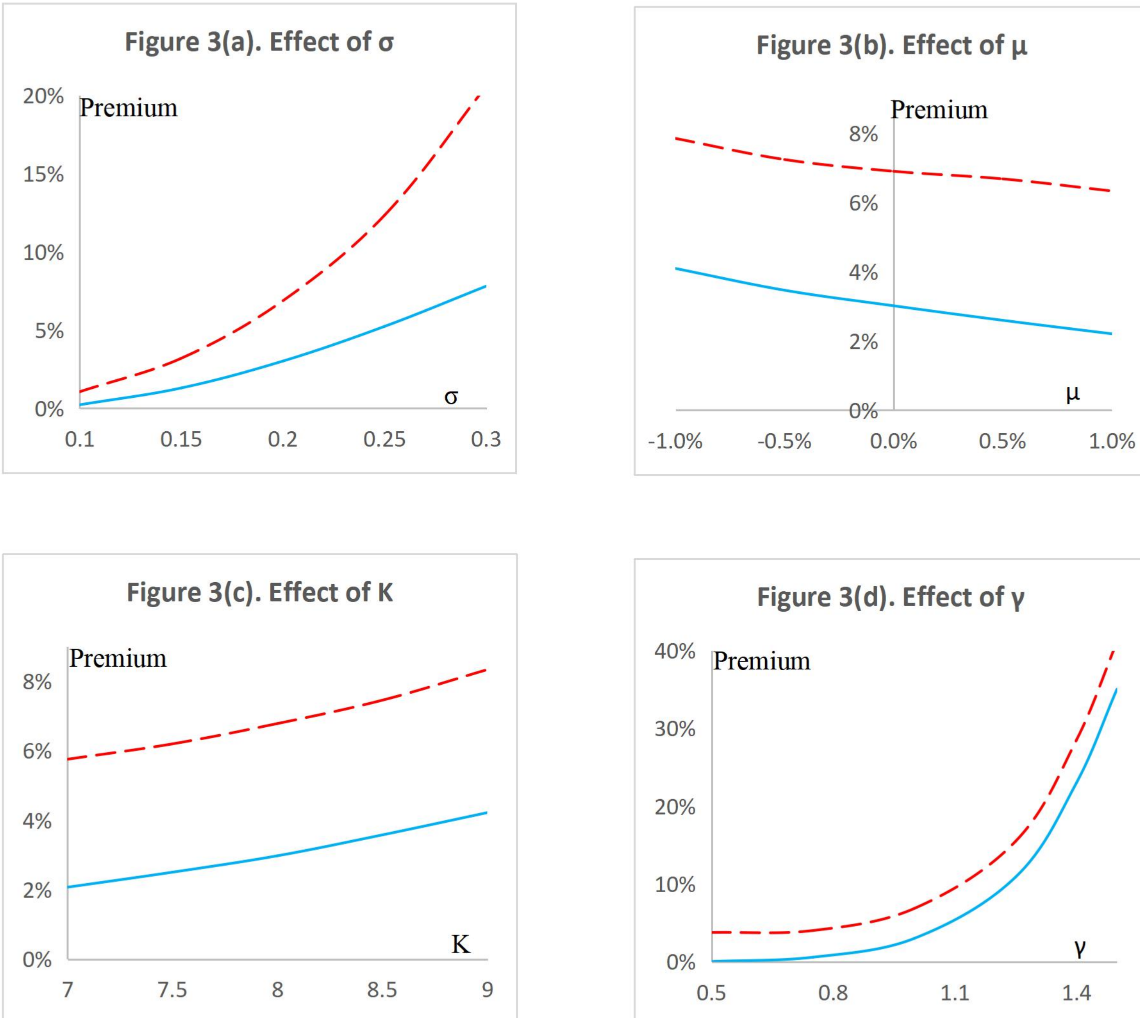
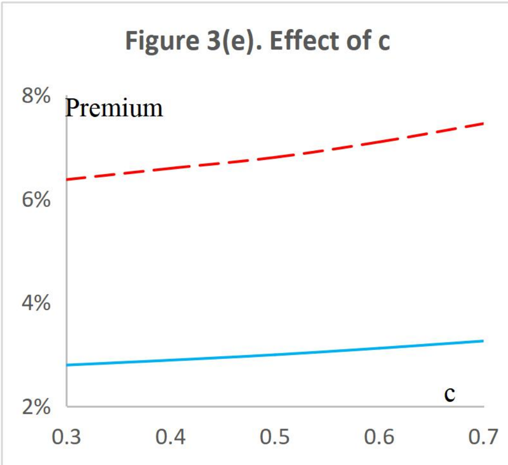
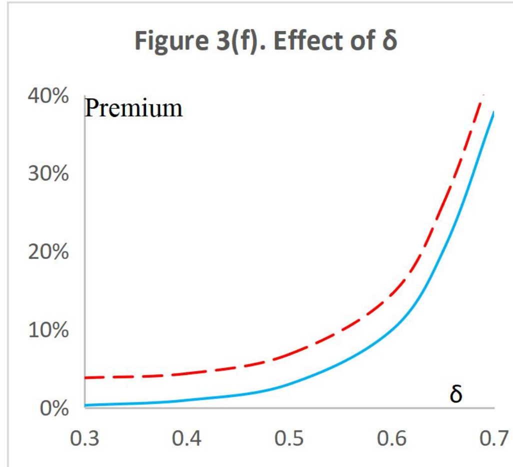


Figure 3. Comparative static results for the premium relative to benchmark firm. The broken red line shows the strategic firm and the solid blue line shows the myopic firm. The base-case parameter values are used: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0$, $m_1 = 5$, K = 8, T = 10, $\theta_0 = 2$, k – 0.2, and $\theta_0 = 6$.





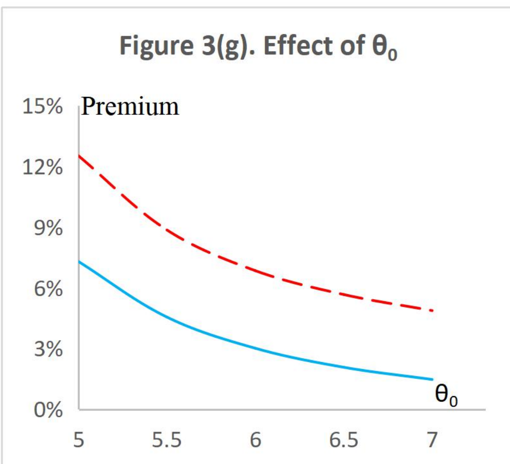


Figure 3. Comparative static results (continued).

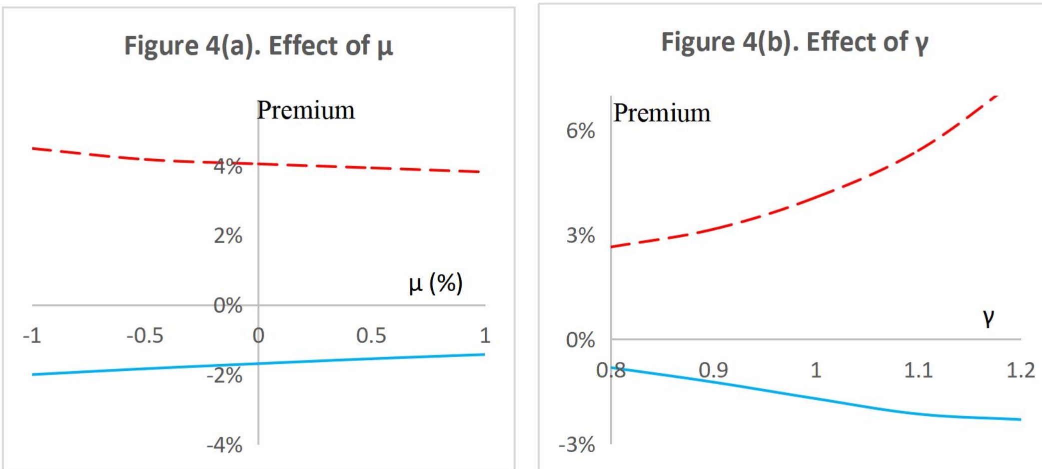


Figure 4. Some comparative static results for the premium *when the myopic firm's premium is negative*. The broken red line shows the strategic firm value and the solid blue line shows the myopic firm value. The base-case parameter values are used: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0$, $m_1 = 5$, $K = 8$, $T = 10$, $\theta_0 = 2$, $k – 0.2$, and $\theta_0 = 6$.

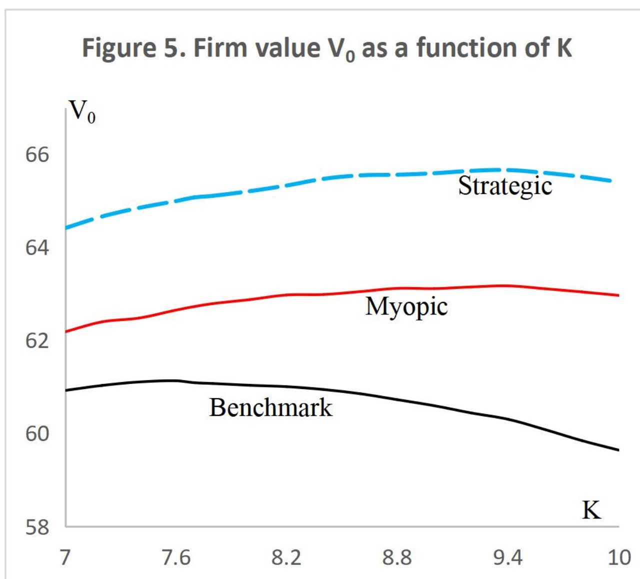


Figure 5. Shows firm value V_0 in all three cases as a function of investment size K, using the base case parameter values: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0$, $m_1 = 5$, $T = 10$, $\theta_0 = 2$, and $\theta_0 = 6$. The optimal investment size is 7.6 for the benchmark firm and 9.4 for both the myopic firm and the strategic firm.

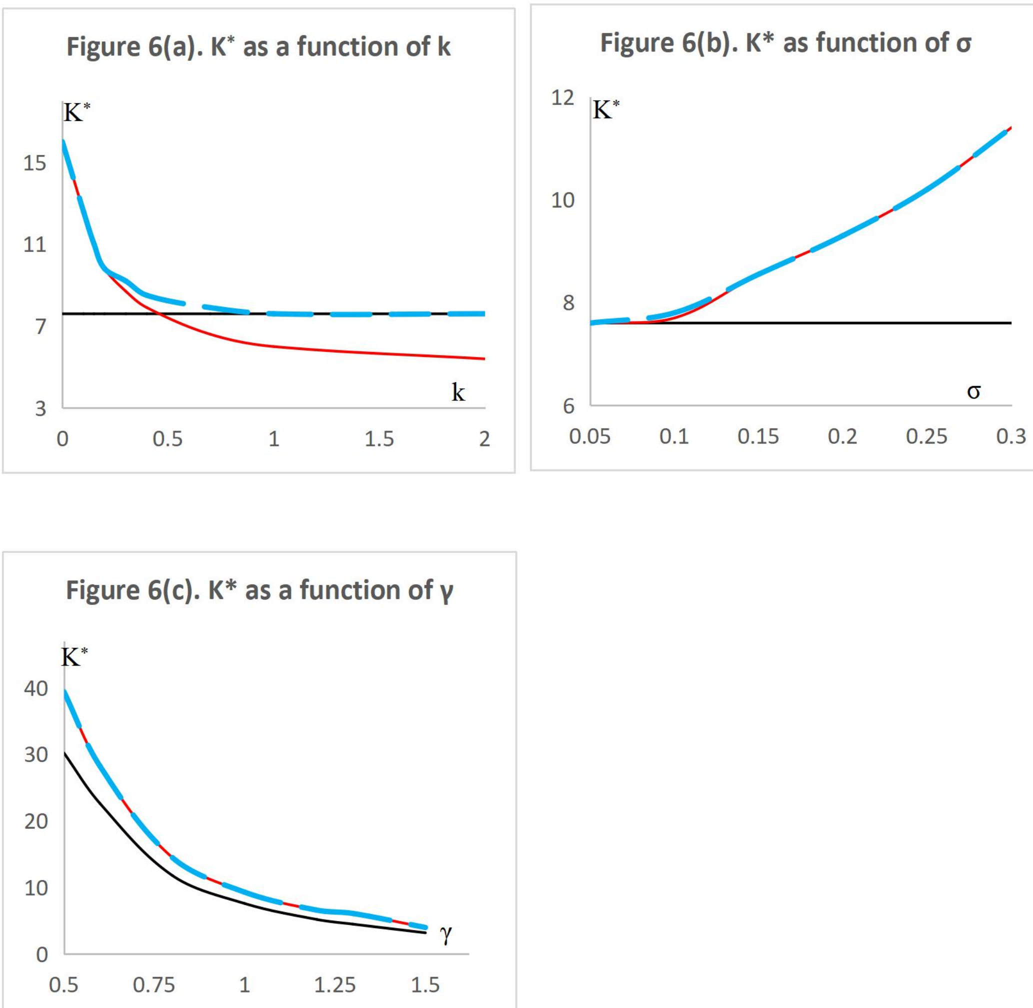


Figure 6. Shows optimal investment size K^* as a function of inventory holding cost k (part a), demand volatility σ (part b) and demand elasticity γ (part c). The base case parameter values are used: $c = 0.5$, $\gamma = 1$, $r = 0.04$, $\mu = 0$, $\sigma = 0.2$, $\delta = 0.5$, $m_0 = 0$, $m_1 = 5$, $T = 10$, $\theta_0 = 2$, and $\theta_0 = 6$.

Appendix A: Valuation of myopic firm

In the simulations, we need to transform previous continuous time modelling of profit and inventory dynamics to a discrete-time setting the following discrete approximations to equations (1):

$$\theta_{i,j} = \theta_{i,j-1} exp\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_{i,j}$$

here the subscript *i* represents the i^th path and *j* denotes the time period the profit flow thereby can be rewritten as

$$\pi_{i,j} = \left(\theta_{i,j} - \gamma q_{i,j}\right)q_{i,j} - cQ - kN_{i,j}$$

The current inventory $N_{i,j}$ can be expressed as

$$N_{i,j} = N_{i,j-1} + \left(Q - q_{i,j}\right)\Delta t$$

Note that $N_{i,j-1}$ is the inventory stock at previous period.

The instantaneous profit at time spot *j* can be rewritten as

$$\pi_{i,j} = \left(\theta_{i,j} - \gamma q_{i,j}\right)q_{i,j} - cQ - k\left(N_{i,j-1} + \left(Q - q_{i,j}\right)\Delta t\right)$$

with following nonnegative requirement

$$\min\left(N_{i,j}, N_{i,j-1}\right) \geq 0$$

The firm value will be calculated by

$$V_{i,j} = \exp\left(-r\Delta t\right) V_{i,j+1} + \int_0^{\Delta t} \pi_{i,j}$$

The last item

$$\int_0^{\Delta t} \pi_{i,j} = \left(\left(\theta_{i,j} - \gamma q_{i,j}\right) q_{i,j} - cQ - kN_{i,j-1}\right) \Delta t - \int_0^{\Delta t} \left(k \int_0^{\boldsymbol{x}} \left(Q - q_{i,j}\right) d\tau\right) d\boldsymbol{x}$$

And it can be further written as

$$\int_0^{\Delta t} \pi_{i,j} = \left(\left(\theta_{i,j} - \gamma q_{i,j}\right) q_{i,j} - cQ - kN_{i,j-1}\right) \Delta t - \frac{1}{2} k\left(Q - q_{i,j}\right) \Delta t^2$$

Note the last integrand captures the cumulative inventory storage cost over Δt.

Recall that the profit flow over Δt is

$$\int_0^{\Delta t} \left(\pi_j \middle| N_{j-1}\right) = \left(\left(\theta_j - \gamma q_j\right) q_j - cQ - kN_{j-1}\right) \Delta t - \frac{1}{2} k\left(Q - q_j\right) \Delta t^2$$

Take F.O.C with respect to q_j we have optimal sales for current time spot j

$$q_j = \frac{2\theta_j + k\Delta t}{4\gamma}$$

The upper sales satisfies $\bar{q}_j \Delta t < N_{j-1} + Q\Delta t$, or we have

$$q_j = \begin{cases} \dfrac{2\theta_j + k\Delta t}{4\gamma} & if\ \ \theta_j < \bar{\theta}_j \\ \dfrac{N_{j-1}}{\Delta t} + Q & if\ \ \theta_j \geq \bar{\theta}_j \end{cases}$$

here $\bar{\theta}_j$ is the upper demand threshold

$$\bar{\theta}_j = 2\gamma\left(\frac{N_{j-1}}{\Delta t} + Q\right) - \frac{k\Delta t}{2}$$

Appendix B: Valuation of strategic firm

The algorithm for strategic firm is more complex due to the unknown inventory decisions.

Before presenting optimal solutions to the entire time series in a simulated discrete time periods, we start with presenting a very simple case. We assume the entire life only has four periods: the demand shocks are $\theta = \theta_t$, for t = 0, 1, 2, 3, 4. Note that production begins at $t = 1$. The demand shocks in the time series will be produced in a randomness generator in MATLAB®

The firm value, as function of sales at three periods q_1, q_2, q_3, q_4 are

$$V_0 = \max_{q_1,q_2,q_3,q_4} (e^{-r\Delta t}\Pi_1 + e^{-r2\Delta t}\Pi_2 + e^{-r3\Delta t}\Pi_3 + e^{-r4\Delta t}\Pi_4)$$

here $\Pi_j = \int_0^{\Delta t} \pi_j$ represents cumulative profit flow over period of Δt

The firm value can be expended easily in the following for a general case, with subscript j as the j^{th} time period

$$V_0 = \max_{q_{j=1,2,3\ldots,T}} \sum_{j=1}^{T} \{e^{-rj\Delta t}[(\theta_j - \gamma q_j)q_j - cQ - kN_{j-1}]\Delta t - k(Q - q_j)\Delta t^2/2\}$$

We list for up to four periods for the purpose of *induction and deduction*. In what follows, we first present detailed solution to the reduced case then we go to implementation details in general case.

$$\begin{aligned} V_0 = e^{-r\Delta t}&\{[(\theta_1 - \gamma q_1)q_1 - cQ]\Delta t - k(Q - q_1)\Delta t^2/2\} \\ &+ e^{-r2\Delta t}\{[(\theta_2 - \gamma q_2)q_2 - cQ - kN_1]\Delta t - k(Q - q_2)\Delta t^2/2\} \\ &+ e^{-r3\Delta t}\{[(\theta_3 - \gamma q_3)q_3 - cQ - kN_2]\Delta t - k(Q - q_3)\Delta t^2/2\} \\ &+ e^{-r4\Delta t}\{[(\theta_4 - \gamma q_4)q_4 - cQ - kN_3]\Delta t - k(Q - q_4)\Delta t^2/2\} \end{aligned}$$

Note that inventory at $t = 0$ is zero, and we need following constraints

$$N_1 = \max(Q - q_1, 0)\Delta t$$

$$N_2 = \max[N_1 + (Q - q_2)\Delta t, 0]$$

$$N_3 = \max[N_2 + (Q - q_3)\Delta t, 0]$$

$$N_4 = \max[N_3 + (Q - q_4)\Delta t, 0]$$

They will ensure nonnegative inventory conditions and regulates the maximum sales at each time. This is a super-large (particularly for full times) constrained optimization problem. Even for the four period case, a traditional optimization such as *Kuhn-Tucker method* with regard to all of q_1, q_2, q_3, q_4, N_1, N_2, N_3, N_4, are difficult to implement here because the max-type function makes the entire value non-differentiable at all domains and it depends on inventory status. We therefore adopt an iteration method, overall our optimization for each simulated path can simply written as

$$\max_{\boldsymbol{q_1,..,q_T,Q}} V_0 = \sum_{i=0}^{T} e^{-ri\Delta t} \Pi_i\big(q_i, N_{i-1}(q_1, .., q_{i-1})\big),$$

$$\text{subject to } \; N_{i=\boldsymbol{Q},i\in[1,T]} > 0 \; \text{ and}$$

$$N_{i\neq\boldsymbol{Q},i\in[1,T]} = 0$$

where, $\boldsymbol{Q}$ represents the collection of all non-zero inventories. Note the challenge here is that the set $\boldsymbol{Q}$ is unknown *ex ante* in the constraints and it has to be solved along with the optimization problem, and we call this *iterative optimization*.

In particular, at initial production $t = 1$, the sales must less or equal than capacity, e.g. $q_1 \leq Q$. Define inventory status as binary: zero or nonzero. We could totally have $2^3 = 8$ scenarios. Notice that the last period inventory N_4 has no impact here. In what follows we only discuss five sample cases for the sake of exhibition. All other cases can be derived in a similar way:

Scenario 1: $N_1 > 0, N_2 > 0, N_3 > 0$

The FOC condition with respect to each sales are as following in the time series

$$\frac{\partial V_0}{\partial q_1} = e^{-r\Delta t}(\theta_1 - 2\gamma q_1)\Delta t + (e^{-r\Delta t}/2 + e^{-r2\Delta t} + e^{-r3\Delta t} + e^{-r4\Delta t})k\Delta t^2 = 0$$

$$\frac{\partial V_0}{\partial q_2} = e^{-2r\Delta t}(\theta_2 - 2\gamma q_2)\Delta t + (e^{-2r\Delta t}/2 + e^{-r3\Delta t} + e^{-r4\Delta t})k\Delta t^2 = 0$$

$$\frac{\partial V_0}{\partial q_3} = e^{-r3\Delta t}(\theta_3 - 2\gamma q_3)\Delta t + (e^{-3r\Delta t}/2 + e^{-r4\Delta t})k\Delta t^2 = 0$$

$$\frac{\partial V_0}{\partial q_4} = e^{-r4\Delta t}((\theta_4 - 2\gamma q_4)\Delta t + k\Delta t^2/2) = 0$$

Then we have

$$q_1^* = \frac{\theta_1 + \left(\frac{1}{2} + e^{-r\Delta t} + e^{-r2\Delta t} + e^{-r3\Delta t}\right) k\Delta t}{2\gamma}$$

$$q_2^* = \frac{\theta_2 + \left(\frac{1}{2} + e^{-r\Delta t} + e^{-r2\Delta t}\right) k\Delta t}{2\gamma}$$

$$q_3^* = \frac{\theta_3 + \left(\frac{1}{2} + e^{-r\Delta t}\right) k\Delta t}{2\gamma}$$

$$q_3^* = \min\left(\frac{\theta_4 + k\Delta t/2}{2\gamma}, Q + \frac{N_3}{\Delta t}\right)$$

There we have optimal sales at each instant time $q_j^* = \frac{\theta_j + \left(\sum_{i=j}^{T} e^{-r(T-i)\Delta t} - \frac{1}{2}\right) k\Delta t}{2\gamma}$, the idea is simple: each optimal sales equals demand shock plus all current and future savings on the inventory storage. Moreover, it is interesting that each optimal sales are independent of future demand shocks. However, the precondition is that inventories at all periods should be positive, that is,

$$N_j = N_{j-1} + \left(Q - q_j\right)\Delta t > 0$$

Actually this condition also equivalently regulates the upper sales quantities

$$\bar{q}_j \Delta t < N_{j-1} + Q\Delta t$$

Or upper demand threshold beyond which the firm will clear inventories, which is obtained by substituting previous solutions on the optimal sales

$$\bar{\theta}_j < 2\gamma\left(\frac{N_{j-1}}{\Delta t} + Q\right) - \left(\sum_{i=j}^{T} e^{-r(T-i)\Delta t} - \frac{1}{2}\right) k\Delta t$$

For simplicity we will not discuss the negative profits, that will generate lower sales (demand) boundaries $\underline{q}_j\left(\underline{\theta}_j\right)$ below which the sales will be zero. In fact, it is doable, e.g. substitute the optimal sales solution back to profit function and solve the positive root because the quadratic equation will be convex shaped. However the multiple switching thresholds will make our algorithm very messy. In what follows we discuss when the precondition of all positive inventories is violated:

Scenario 2: $N_1 = N_2 = N_3 = 0$

$q_1^* = q_2^* = q_3^* = Q$, q_4^* is same as in Case I

Scenario 3: $N_1 = 0, N_2 > 0, N_3 > 0$

$q_1^* = Q$, q_2^*, q_3^*, q_4^* are same as in Case I,

Scenario 4: $N_1 > 0$, $N_2 = 0$, $N_3 > 0$

This scenario is more complicated. Because when $N_2 = 0$, it means we couldn't freely optimize q_2^*, instead, q_2^* is constrained by $q_2^* = 2Q - q_1^*$. So this case becomes constrained optimization problem.

$$maxV_0(q_1, .., q_4), \text{ subject to } 2Q - q_1 - q_2 = 0$$

So we have to resubstitute the updated q_2^* to optimize q_1^*:

$$q_1^* = \frac{\theta_1 - e^{-r\Delta t}\theta_2 + 4\gamma Q e^{-r\Delta t} + k\Delta t(1 + e^{-r\Delta t})/2}{2\gamma(1 + e^{-r\Delta t})}$$

This equation can be further rewritten as

$$q_1^* = \underbrace{\frac{\theta_1 + k\Delta t/2}{2\gamma}}_{\boldsymbol{myopic\ effect}} + \underbrace{\left(2Q - \frac{\theta_1 + \theta_2}{2\gamma}\right)\frac{e^{-r\Delta t}}{1 + e^{-r\Delta t}}}_{\boldsymbol{inventory\ effect}}$$

This equation has two parts: the first part is same as myopic case. The second part captures inventory effect: e.g. when the future demand θ_2 increases, then q_1^* decreases, that is, the previous sales should decrease to leave some inventory for future sales, which is intuitive since future price will become higher. The affiliated item $\frac{e^{-r\Delta t}}{1+e^{-r\Delta t}}$ can be considered as the “weight ratio” to capture the discount weight of future effect. Last, q_3^* and q_4^* are same as in Case I

Scenario 5: $N_1 > 0$, $N_2 > 0$, $N_3 = 0$

The logic is similar to the Scenario 4. We have freedom to optimally select q_1^* and q_2^* while leave equality constraint $q_3^* = 3Q - q_1^* - q_2^*$. We substitute this constrained to the value function and take first derivative to both q_1 and q_2:

$$q_2^* = \frac{\theta_2 + k\Delta t/2}{2\gamma} + \left(3Q - q_1^* - \frac{\theta_2 + \theta_3}{2\gamma}\right)\frac{e^{-r\Delta t}}{1 + e^{-r\Delta t}}$$

and at the first period

$$q_1^* = \frac{\theta_1 + g(k\Delta t/2)}{2\gamma} + \left(3Q - q_2^* - \frac{\theta_1 + \theta_3}{2\gamma}\right)\frac{e^{-2r\Delta t}}{1 + e^{-2r\Delta t}}$$

here the coefficient

$$\mathcal{g} = 1 + \frac{2e^{-r\Delta t}}{1 + e^{-2r\Delta t}}$$

Obviously we can solve the two-variable equations for q_1^* and q_2^*, however, when there are many periods until inventory encounters zero at the first time (e.g., the constraint becomes $q_j^* = iQ - \sum_{z=1}^{i-1} q_{z\neq j}^*$) we can't do this. For example, suppose $N_1, N_2, N_3, ..., N_{i-1} > 0$ and $N_i = 0$ and let's derive q_1^* , q_2^*, q_3^*,…, q_{i-1}^*:

$$q_j^* = \frac{\theta_j + \mathcal{g}_j(k\Delta t/2)}{2\gamma} + \left(iQ - \sum_{z=1}^{i-1} q_{z\neq j}^* - \frac{\theta_j + \theta_i}{2\gamma}\right)\frac{e^{-(i-j)r\Delta t}}{1 + e^{-(i-j)r\Delta t}}$$

The coefficient $\mathcal{g}_j$ can be expressed as

$$\mathcal{g}_j = \begin{cases} 1 + \dfrac{2\sum_{v=1}^{i-2} e^{-vr\Delta t}}{1 + e^{-(i-j)r\Delta t}} & for\ \ j > i-1 \\ 1 & for\ \ j = i-1 \end{cases}$$

The equation system can be rewritten as

$$\boldsymbol{A}_{(\boldsymbol{i}-\mathbf{1})\times(\boldsymbol{i}-\mathbf{1})}\boldsymbol{q}^*_{(\boldsymbol{i}-\mathbf{1})\times\mathbf{1}} = \boldsymbol{B}_{(\boldsymbol{i}-\mathbf{1})\times\mathbf{1}}$$

where the bold matrices are as following

$$\boldsymbol{q}^*_{(\boldsymbol{i}-\mathbf{1})\times\mathbf{1}} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_{i-1} \end{bmatrix}$$

$$\boldsymbol{B}_{(\boldsymbol{i}-\mathbf{1})\times\mathbf{1}} = \begin{bmatrix} \dfrac{\theta_1 + \mathcal{g}_1(k\Delta t/2)}{2\gamma} + \left(iQ - \dfrac{\theta_1 + \theta_i}{2\gamma}\right)\dfrac{e^{-(i-1)r\Delta t}}{1 + e^{-(i-1)r\Delta t}} \\ \dfrac{\theta_2 + \mathcal{g}_2(k\Delta t/2)}{2\gamma} + \left(iQ - \dfrac{\theta_2 + \theta_i}{2\gamma}\right)\dfrac{e^{-(i-2)r\Delta t}}{1 + e^{-(i-2)r\Delta t}} \\ \vdots \\ \dfrac{\theta_{i-1} + \mathcal{g}_{i-1}(k\Delta t/2)}{2\gamma} + \left(iQ - \dfrac{\theta_{i-1} + \theta_i}{2\gamma}\right)\dfrac{e^{-r\Delta t}}{1 + e^{-r\Delta t}} \end{bmatrix}$$

$$
\boldsymbol{A}_{(\boldsymbol{i}-\boldsymbol{1})\times(\boldsymbol{i}-\boldsymbol{1})} = \begin{bmatrix} 1 & a_1 & a_1 & \cdots & a_1 \\ a_2 & 1 & a_2 & \cdots & a_2 \\ a_3 & a_3 & \ddots & a_3 & a_3 \\ \vdots & a_4 & a_4 & 1 & \vdots \\ a_{i-1} & a_{i-1} & a_{i-1} & \cdots & 1 \end{bmatrix}
$$

here $a_j = \frac{e^{-(i-j)r\Delta t}}{1+e^{-(i-j)r\Delta t}}$ are the last coefficient in the equation. To solve the linear system we use the Gauss-Seidel method.

Finally, suppose we simulate *M* paths, the above calculation will be repeated *M* times. This simple 3-period case informs some intuitions:

1. The optimal sales (or inventory) level at current time *t* depends on both previous and future optimal sales (or inventory) level, which should be difficult to produce a closed form solution
2. Luckily, all future storing costs of optimal sales expression will cut off at the first timing of zero inventories.
3. the solutions to optimal sales can be converted to solutions to determine the optimal timing of depleting inventories, which may still be impossible to solve (for N time periods we will have 2^N cases of inventories status (e.g. zero or positive)).

To solve the problem we propose a reduced reinforced learning algorithm. In fact, our idea is largely similar to the policy iteration used in classic dynamic programming problems but our PDE equation is relatively implicit, for instance, it lacks of explicit expression of instant optimal sales and appropriate boundary conditions. Thus traditional policy iteration algorithm couldn't applied easily here. The general algorithm can be described as follows:

1. To start with, we assume all periods of inventories are positive, and compute the corresponding optimal sales, using equation(X), then we recalculate the updated inventories, if all of them are positive, it is one of our solutions, if not, go to step 2:

2. Suppose the first period of zero inventory is t = i, then we check if inventories of all previous periods, e.g. $t = 2, 3, \dots, i - 1$, are positive. If not then move the search to $t = i + 1$ and redo this part, if yes, then do follows:

   2.1 Assume all inventories after t = i are positive, and update if under such assumption both $N_i = 0$, and $N_{i+1,\dots T} > 0$ (e.g. all assumptions are valid). If yes, we find the solution. If not, we do next:

   2.2 then we will have two possible outcomes: (1) if $N_i > 0$, then our assumption on $N_i = 0$ is invalid and we move a new search to $t = i + 1$, the reason is that the assumption of all positive future inventories has maximized q_i (e.g. minimize N_i) (2) if $N_i = 0$, but not all updated future inventories are positive then it indicates there should be a second period, $t = j$ ($j > i$), at which $N_j = 0$, e.g. repeat step 2 to find out *j* period.

3. Note that Step 2 could generate multiple solutions for $N_i = 0$ or/and $N_j = 0$, given $max(i, j) < T$. Thus we need to repeat both steps to find all possible solutions until the end of time series. To facilitate this process, we can firstly find the latest time when inventory becomes zero. Then all previous periods should also be candidates (see equation (xx))

Although to prove the convergence and stability of our algorithm is beyond our capacity, it should not be hard to convince, since we will have a finite set of solutions, e.g. 2^N, there always exists the best set to maximize equity value *ex ante*, in an iterative procedure that always improves value function, we should get the optimal solution in a certain number of iterations.